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A q-deformed osp(1, 2) superalgebra and its two-component coherent state representations

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Abstract. A q-deformed osp(1, 2) superalgebra is defined by the use of a pair of q-boson annihilation and creation operators. A kind of two-component coherent state representation of the q-deformed superalgebra is found. And a q-differential realization of the q-deformed osp(1, 2) superalgebra is obtained.

1. Introduction

During recent years, quantum (super)algebras have made their appearance in an ever-increasing number of problems in physics and mathematics, ranging from non-commutative geometry to integrable systems in statistical mechanics and conformal field theory, and solvable models in molecular and nuclear spectroscopy.

In order to apply quantum (super)algebras in physics, one needs a well-developed theory of their irreducible representations. In the case of ordinary Lie (super)algebras, the boson realization method [1–3] and related coherent state theory [4] have proved very useful for studying representations. Since a q-deformed oscillator was defined in terms of q-boson operators [5], it has been widely used to construct representations of quantum (super)algebras [6–8]. So far the coherent states for a q-deformed oscillator and the $su_q(2)$ algebra have been investigated in detail by several authors [5, 9, 10], and some applications of them in physics have also been exploited [11].

As is well known, the osp(1, 2) superalgebra is one of the most fundamental Lie superalgebras, so it is meaningful to study its q-deformed version. Recently, some papers have been devoted to the universal enveloping algebra of the q-deformed osp(1, 2) superalgebra and its representations [8, 12]. In this paper, we will define a q-deformed osp(1, 2) superalgebra in terms of a pair of q-boson operators, and construct a type of two-component coherent state representations, and propose a q-differential realization for the q-deformed osp(1, 2) superalgebra.

2. A q-deformed osp(1, 2) superalgebra

Let us firstly recall the classical osp(1, 2) superalgebra

$$\begin{aligned} &[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0 \\ &[J_0, V_\pm] = \pm\tfrac{1}{2}V_\pm \qquad [J_\pm, V_\pm] = 0 \qquad [J_\pm, V_\mp] = V_\pm \\ &\{V_\pm, V_\pm\} = \pm\tfrac{1}{2}J_\pm \qquad \{V_+, V_-\} = -\tfrac{1}{2}J_0 \end{aligned} \tag{1}$$

which can be realized in terms of a pair of boson annihilation and creation operators b and b^+ ($[b, b^+]=1$)

$$J_+=-\tfrac{1}{2}b^{+2} \qquad J_-=\tfrac{1}{2}b^2 \qquad J_0=\tfrac{1}{2}b^+b+\tfrac{1}{4}$$
$$V_+=\frac{\mathrm{i}}{2\sqrt{2}}b^+ \qquad V_-=\frac{\mathrm{i}}{2\sqrt{2}}b. \tag{2}$$

In order to define the q-deformed osp(1, 2) superalgebra, we introduce the following q-deformed operators

$$J_+^q=-(q^{1/2}+q^{-1/2})^{-1}a^{+2} \qquad J_-^q=(q^{1/2}+q^{-1/2})^{-1}a^2 \qquad J_0^q=\tfrac{1}{2}(N+\tfrac{1}{2})$$
$$V_+^q=\tfrac{1}{2}\mathrm{i}(q^{1/2}+q^{-1/2})^{-1/2}a^+ \qquad V_-^q=\tfrac{1}{2}\mathrm{i}(q^{1/2}+q^{-1/2})^{-1/2}a \tag{3}$$

where corresponding to b and b^+, respectively, a and a^+ are a pair of q-deformed boson operators and satisfy

$$[N, a]=-a \qquad [N, a^+]=a^+ \qquad aa^+-q^{-1/2}a^+a=q^{N/2}. \tag{4}$$

The operators a and a^+ act in a Hilbert space with basis $|n\rangle$ ($n=0, 1, 2, \ldots$), such that

$$a|n\rangle=\sqrt{[n]}\,|n-1\rangle \qquad a^+|n\rangle=\sqrt{[n+1]}\,|n+1\rangle \qquad |n\rangle=\frac{a^{+n}}{\sqrt{[n]!}}|0\rangle \tag{5}$$

where

$$[x]\equiv\frac{q^{x/2}-q^{-x/2}}{q^{1/2}+q^{-1/2}}. \tag{6}$$

Making use of equations (4) and (5), one can show that the q-operators defined in (3) give rise to a q-deformation of the osp(1, 2) superalgebra

$$[J_0^q, J_\pm^q]=\pm J_\pm^q \qquad [J_+^q, J_-^q]=[2J_0^q]_q$$
$$[J_0^q, V_\pm^q]=\pm\tfrac{1}{2}V_\pm^q \qquad [J_\pm^q, V_\pm^q]=0$$
$$[J_+^q, V_-^q]=(q^{1/4}+q^{-1/4})([J_0^q]_q+[-J_0^q+\tfrac{1}{2}]_q)V_+^q \tag{7}$$
$$[J_-^q, V_+^q]=(q^{1/4}+q^{-1/4})([-J_0^q]_q+[J_0^q+\tfrac{1}{2}]_q)V_-^q$$
$$\{V_\pm^q, V_\pm^q\}=\pm\tfrac{1}{2}J_\pm^q \qquad \{V_+^q, V_-^q\}=\tfrac{1}{4}(q^{1/4}+q^{-1/4})[-J_0^q]_q$$

where

$$[x]_q\equiv\frac{q^x-q^{-x}}{q+q^{-1}}. \tag{8}$$

It is obvious that the q-deformed osp(1, 2) superalgebra becomes the the ordinary osp(1, 2) superalgebra when the deformation parameter $q\to 1$.

3. Two-component coherent state representations

Here we will propose a type of two-component coherent state representations of the q-deformed osp(1, 2) superalgebra. Making use of the q-boson realization of the generators (3), we introduce two states

$$|z\rangle_1 = N_1(z)\left(e_q^{-2i(q^{1/2}+q^{-1/2})^{1/2} z V_+^q} + e_q^{2i(q^{1/2}+q^{-1/2})^{1/2} z V_+^q}\right)|0\rangle$$
$$= N_1(z) \sum_{n=0}^{\infty} \frac{z^{2n}}{\sqrt{[2n]!}} |2n\rangle \tag{9a}$$
$$|z\rangle_2 = N_2(z)\left(e_q^{-2i(q^{1/2}+q^{-1/2})^{1/2} z V_+^q} - e_q^{2i(q^{1/2}+q^{-1/2})^{1/2} z V_+^q}\right)|0\rangle$$
$$= N_2(z) \sum_{n=0}^{\infty} \frac{z^{2n+1}}{\sqrt{[2n+1]!}} |2n+1\rangle \tag{9b}$$

where $e_q^x = \sum_{n=0}^{\infty} x^n/[n]!$, $N_1(z)$ and $N_2(z)$ are normalization constants, and z is a complex number.

The analyses below show that $|z\rangle_1$ together with $|z\rangle_2$ form a kind of coherent state of the q-deformed osp(1, 2) superalgebra, denoted by $\{|z\rangle_1, |z\rangle_2\}$, in which $|z\rangle_1$ and $|z\rangle_2$ may be regarded as two subspaces (two components) of the coherent states.

Making use of the q-boson realization of the generators (3) and (5), one can easily check that $|z\rangle_1$ and $|z\rangle_2$ are eigenstates of the operator J_-^q

$$J_-^q|z\rangle_1 = (q^{1/2}+q^{-1/2})^{-1} z^2 |z\rangle_1 \qquad J_-^q|z\rangle_2 = (q^{1/2}+q^{-1/2})^{-1} z^2 |z\rangle_2. \tag{10}$$

We require that the coherent states are normalized in the form

$${}_i\langle z|z\rangle_i = 1 \qquad (i = 1, 2) \tag{11}$$

then, the normalization constants are given by

$$N_1(z) = \tfrac{1}{2}(\cosh_q(z\bar{z}))^{-1/2} \qquad N_2(z) = \tfrac{1}{2}(\sinh_q(z\bar{z}))^{-1/2} \tag{12}$$

where we have introduced two q-functions

$$\cosh_q x = \tfrac{1}{2}(e_q^x + e_q^{-x}) \qquad \sinh_q x = \tfrac{1}{2}(e_q^x - e_q^{-x}). \tag{13}$$

The coherent states have orthogonality relations

$$\begin{aligned} {}_1\langle z'|z\rangle_1 &= 4N_1(z)N_1(z')\cosh_q(z\bar{z}') \\ {}_2\langle z'|z\rangle_2 &= 4N_2(z)N_2(z')\sinh_q(z\bar{z}') \\ {}_1\langle z'|z\rangle_2 &= 0 \end{aligned} \tag{14}$$

which means that two coherent states in different subspaces are orthogonal each other, but not in the same subspace.

We now find a resolution of unity for the coherent states $\{|z\rangle_1, |z\rangle_2\}$. Since the state vectors $\{|n\rangle, n = 0, 1, 2, \ldots\}$ are known to form a completeness orthonormal set, the problem here may be changed to find the following two weight functions $\sigma_1(z)$ and $\sigma_2(z)$ such that

$$\int d_q\,\sigma_1(z)|z\rangle_{11}\langle z| + \int d_q\,\sigma_2(z)|z\rangle_{22}\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n|$$
$$= I$$

where I is the identity operator.

Let $|f\rangle$ and $|g\rangle$ be two arbitrary vectors, then equation (15) means that

$$\langle f|g\rangle = \int d_q\sigma_1(z)\langle f|z\rangle_{11}\langle z|g\rangle + \int d_q\sigma_2(z)\langle f|z\rangle_{22}\langle z|g\rangle. \tag{16}$$

We now determine the two weight functions. Let

$$d_q\sigma_1(z)=\sigma_1(r)r\,d_qr\,d\theta \qquad d_q\sigma_2(z)=\sigma_2(r)r\,d_qr\,d\theta \tag{17}$$

where we have set $z=r\,e^{i\theta}$, d_qr is a q-differential while $d\theta$ is an ordinary differential.

Substituting (9) and (17) into (16), and integrating over the variable θ from 0 to 2π we have

$$\begin{aligned}\langle f|g\rangle &=\tfrac{1}{4}\sum_{n,m=0}^{\infty}\langle f|2n\rangle\langle 2m|g\rangle\int_0^\infty d_qr\int_0^{2\pi}d\theta\, e^{2i(n-m)\theta}\frac{\sigma_1(r)r^{2(n+m)+1}}{\cosh_q r^2}\\ &\quad+\tfrac{1}{4}\sum_{n,m=0}^{\infty}\langle f|2n+1\rangle\langle 2m+1|g\rangle\int_0^\infty d_qr\int_0^{2\pi}d\theta\, e^{2i(n-m+1)\theta}\frac{\sigma_2(r)r^{2(n+m)+3}}{\sinh_q r^2}\\ &=\frac{\pi}{2[2]}\sum_{n=0}^{\infty}\langle f|2n\rangle\langle 2n|g\rangle\int_0^\infty d_qr^2\frac{\sigma_1(r)r^{4n}}{[2n]!\cosh_q r^2}\\ &\quad+\frac{\pi}{2[2]}\sum_{n=0}^{\infty}\langle f|2n+1\rangle\langle 2n+1|g\rangle\int_0^\infty d_qr^2\frac{\sigma_2(r)r^{4n+2}}{[2n+1]!\sinh_q r^2}.\end{aligned} \tag{18}$$

Hence, we must have

$$\frac{\pi}{2[2]}\int_0^\infty d_qr^2\frac{\sigma_1(r)r^{4n}}{[2n]!\cosh_q r^2}=1 \qquad \frac{\pi}{2[2]}\int_0^\infty d_qr^2\frac{\sigma_2(r)r^{4n+2}}{[2n+1]!\sinh_q r^2}=1. \tag{19}$$

With the help of techniques of q-analysis, we can find

$$\sigma_1(r)=\frac{2[2]}{\pi}e_q^{-r^2}\cosh_q r^2 \qquad \sigma_2(r)=\frac{2[2]}{\pi}e_q^{-r^2}\sinh_q r^2. \tag{20}$$

Therefore, the resolution of unity for the coherent states $\{|z\rangle_1, |z\rangle_2\}$ can be expressed as

$$\frac{2}{\pi}\iint d_qr^2\,d\theta\, e_q^{-r^2}\{\cosh_q r^2|r\,e^{i\theta}\rangle_{11}\langle r\,e^{i\theta}|+\sinh_q r^2|r\,e^{i\theta}\rangle_{22}\langle r\,e^{i\theta}|\}=I. \tag{21}$$

As a result of the above completeness relation, an arbitrary vector $|\psi\rangle$ can be expanded in terms of the coherent states for the q-deformed osp(1, 2) superalgebra as follows:

$$|\psi\rangle=\frac{2}{\pi}\iint d_qr^2\,d\theta\, e_q^{-r^2}\{\cosh_q r^2|r\,e^{i\theta}\rangle_{11}\langle r\,e^{i\theta}|\psi\rangle+\sinh_q r^2|r\,e^{i\theta}\rangle_{22}\langle r\,e^{i\theta}|\psi\rangle\}. \tag{22}$$

4. A q-differential realization of the q-deformed osp(1, 2) superalgebra

In this section, we shall present a q-differential realization of the q-deformed osp(1, 2) superalgebra. For simplicity, we consider a q-differential realization in the unnormalized coherent state space $\{\|z\rangle_1, \|z\rangle_2\}$ defined by

$$\|z\rangle_1=\sum_{n=0}^{\infty}\frac{z^{2n}}{\sqrt{[2n]!}}|2n\rangle \qquad \|z\rangle_2=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{\sqrt{[2n+1]!}}|2n+1\rangle. \tag{23}$$

one can find easily the expansion coefficients

$$\begin{aligned}&\langle 2n|z\rangle_1=\frac{z^{2n}}{\sqrt{[2n]!}} \qquad &&\langle 2n+1|z\rangle_1=0\\ &\langle 2n|z\rangle_2=0 &&\langle 2n+1|z\rangle_2=\frac{z^{2n+1}}{\sqrt{[2n+1]!}}.\end{aligned} \tag{24}$$

We now consider the actions of the genrators of the q-deformed osp(1, 2) superalgebra on $\{\|z\rangle_1, \|z\rangle_2\}$. Firstly, we calculate the action of J_+^q:

$$\begin{aligned} J_+^q\|z\rangle_1 &= -\sum_{n=0}^{\infty}(q^{1/2}+q^{-1/2})^{-1}a^{+2}\{|2n\rangle\langle 2n|+|2n+1\rangle\langle 2n+1|\}\|z\rangle_1 \\ &= -\sum_{n=0}^{\infty}(q^{1/2}+q^{-1/2})^{-1}a^{+2}|2n\rangle\langle 2n\|z\rangle_1 \\ &= -\sum_{n=0}^{\infty}(q^{1/2}+q^{-1/2})^{-1}\left[\frac{[2n+1][2n+2]}{[2n]!}\right]^{1/2}z^{2n}|2n+2\rangle \\ &= -(q^{1/2}+q^{-1/2})^{-1}\sum_{n'=0}^{\infty}\frac{[2n'][2n'-1]}{\sqrt{[2n']!}}|2n'\rangle \\ &= -(q^{1/2}+q^{-1/2})^{-1}\frac{\mathrm{d}^2}{\mathrm{d}_q z^2}\|z\rangle_1 \end{aligned} \tag{25a}$$

which indicates that the generator J_+^q acts like a q-differential operator

$$-(q^{1/2}+q^{-1/2})^{-1}\frac{\mathrm{d}^2}{\mathrm{d}_q z^2}$$

on the subspace $\{\|z\rangle_1\}$. In the same way, one can obtain the action of J_+^q on the second subspace $\{\|z\rangle_2\}$:

$$J_+^q\|z\rangle_2 = -(q^{1/2}+q^{-1/2})^{-1}\frac{\mathrm{d}^2}{\mathrm{d}_q z^2}\|z\rangle_2. \tag{25b}$$

Then, the action of the generator J_+^q on $\{\|z\rangle_1, \|z\rangle_2\}$ may be expressed as

$$J_+^q\begin{pmatrix}\|z\rangle_1\\ \|z\rangle_2\end{pmatrix} = \rho(J_+^q)\begin{pmatrix}\|z\rangle_1\\ \|z\rangle_2\end{pmatrix} \tag{26}$$

where

$$\rho(J_+^q) = -(q^{1/2}+q^{-1/2})^{-1}\begin{pmatrix}\dfrac{\mathrm{d}^2}{\mathrm{d}_q z^2} & 0\\ 0 & \dfrac{\mathrm{d}^2}{\mathrm{d}_q z^2}\end{pmatrix}. \tag{27}$$

Similarly, one can get the actions of the other generators on $\{\|z\rangle_1, \|z\rangle_2\}$:

$$\rho(J_-^q) = (q^{1/2}+q^{-1/2})^{-1}\begin{pmatrix} z^2 & 0\\ 0 & z^2\end{pmatrix} \qquad \rho(J_0^q) = \tfrac{1}{2}\begin{pmatrix} z\dfrac{\mathrm{d}}{\mathrm{d}_q z}+\tfrac{1}{2} & 0\\ 0 & z\dfrac{\mathrm{d}}{\mathrm{d}_q z}+\tfrac{1}{2}\end{pmatrix} \tag{28}$$

$$\rho(V_+^q) = \tfrac{1}{2}\mathrm{i}(q^{1/2}+q^{-1/2})^{-1}\begin{pmatrix} 0 & \dfrac{\mathrm{d}}{\mathrm{d}_q z}\\ \dfrac{\mathrm{d}}{\mathrm{d}q_z} & 0\end{pmatrix} \qquad \rho(V_-^q) = \tfrac{1}{2}\mathrm{i}(q^{1/2}+q^{-1/2})^{-1}\begin{pmatrix} 0 & z\\ z & 0\end{pmatrix} \tag{29}$$

It is straightforward to verify that these matrix q-differential operators in (27), (28) and (29) satisfy the commutation and anti-commutation relations of the q-deformed osp(1, 2) superalgebra, so they give rise to a q-differential realization of the q-deformed superalgebra.

5. Concluding remarks

We have defined a q-deformed osp(1, 2) superalgebra in terms of one pair of q-boson annihilation and creation operators, and constructed two-component coherent state representation of the q-deformed superalgebra. It should be mentioned that the two-component coherent states for the q-deformed osp(1, 2) superalgebra are, happily, q-analogues of the even and odd coherent states [13, 14], which have important applications in quantum optics. We have also obtained a q-differential of the q-deformed osp(1, 2) superalgebra. It is interesting to note that in this q-differential we have used only one complex and a q-differential operator without any Grassmann variables.

It is interesting to exploit further applications of the two-component coherent states for the q-deformed osp(1, 2) superalgebra in nonlinear optics.

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